Electromagnetism

Contents

1 Electrostatics

Here we define

$$\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{r}' \qquad \mathscr{r} = |\mathbf{r} - \mathbf{r}'|$$

where $\mathbf{r}$ is the point of interest, and $\mathbf{r}'$ is the position of the source charge.

1.1 Coulomb's Law

For a point charge q at $\mathbf{r}'$, the electric field is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

For a continuous charge distribution,

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\,d\tau'$$

1.2 Gauss's Law

Gauss's law is derived by integrating the flux through a closed surface.

$$\oint_S \mathbf{E}\cdot d\mathbf{a} = \frac{Q_{\text{in}}}{\epsilon_0}$$

With divergence theorem, which is

$$\int \boldsymbol{\nabla}\cdot\mathbf{E}\,d\tau = \oint_S \mathbf{E}\cdot d\mathbf{a}$$

we get

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$$

1.3 Divergence and Curl of E

The divergence of $\mathbf{E}$ is given by Gauss's law. For the curl, we find that the integral of $\mathbf{E}$ is 0,

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0$$

Using Stokes' theorem, which is

$$\int (\boldsymbol{\nabla}\times\mathbf{E})\cdot d\mathbf{a} = \oint \mathbf{E}\cdot d\mathbf{l}$$

we get the curl of $\mathbf{E}$,

$$\boldsymbol{\nabla}\times\mathbf{E} = \mathbf{0}$$

1.4 Potential

If we define $V = 0$ at infinity, then

$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}$$

which is

$$V(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

Using the fundamental theorem for gradient, which is

$$V(\mathbf{b}) - V(\mathbf{a}) = \int_{\mathbf{a}}^{\mathbf{b}} \boldsymbol{\nabla} V \cdot d\mathbf{l}$$

we get

$$\mathbf{E} = -\boldsymbol{\nabla} V$$

Rewrite $\boldsymbol{\nabla} \cdot \mathbf{E}$ in terms of V, we get Poisson's equation,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

and Laplace's equation,

$$\nabla^2 V = 0$$

1.5 Energy

The energy of a charge distribution is given by

$$W = \frac{1}{2} \int \rho V d\tau = -\frac{\epsilon_0}{2} \int V \nabla^2 V d\tau$$

Integration by part, and using the fact that $V = 0$ at ∞, we get

$$W = \frac{\epsilon_0}{2} \int E^2 d\tau$$

where the integration is over all space.
Notice that the energy can be interpreted as the consequence of Coulomb's force, but we can also think of it as the energy stored in the **E** field surrounding the charges. The energy density of the **E** field is thus

$$w = \frac{\epsilon_0}{2} E^2$$

Notice that from w the energy is always non-negative. This is because w contains the "self-energy" contributions of charges, while the previous treatment does not.

1.6 Conductors

An ideal conductor has the following properties,

1. $\rho = 0$ inside the conductor,
2. any net charge resides on the surface,
3. it is equipotential (including surface),
4. near the surface, **E** is perpendicular to the surface.

1.7 More on Potential

We could use multipole expansion to approximate a potential. The basic idea is to expand V using a Taylor expansion. We have

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')}{\mathscr{r}} d\tau'$$

When we are far from the distribution, we can expand the $1/\mathscr{r}$ term. This will yield the multipole expansion.

The dipole term is of great importance. It is given by

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' \cos\alpha \rho(\mathbf{r}') d\tau'$$

where α is the angle between **r** and **r**′. We difine the dipole moment of the charge distribution to be

$$\mathbf{p} = \int \mathbf{r}' \rho(\mathbf{r}') d\tau'$$

Then we can write

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{P} \cdot \hat{\mathbf{r}}}{r^2}$$

2 Electric Fields in Matter

The matter we will discuss is dielectrics, whose nickname is insulators.

2.1 Polarization

Polarization $\mathbf{P}$ is defined to be the dipole moment per unit volume. So the following relation holds,

$$\mathbf{p} = \int \mathbf{P} d\tau$$

The potential of a dipole now becomes

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{P} \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'$$

Note that in the integral r is replaced by $\mathscr{r}$. This is because $\mathbf{P}$ is not relative to one particular point, but relative to its $d\tau'$. So we have to treat the distribution as infinitely many small dipoles rather than one dipole. We have

$$\boldsymbol{\nabla}' \left(\frac{1}{\mathscr{r}} \right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

So

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \mathbf{P} \cdot \boldsymbol{\nabla}' \left(\frac{1}{\mathscr{r}} \right) d\tau'$$

Vector identities and divergence theorem gives

$$V_{\text{dip}}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oint \frac{\mathbf{P}}{\mathscr{r}} \cdot d\mathbf{a} - \frac{1}{4\pi\epsilon_0} \int \frac{1}{\mathscr{r}} (\boldsymbol{\nabla}' \cdot \mathbf{P}) d\tau'$$

Comparing the equation to potential expressions, we get

$$\sigma_b = \mathbf{P} \cdot \hat{\mathbf{n}} \qquad \rho_b = -\boldsymbol{\nabla}' \cdot \mathbf{P}$$

where subscript b stands for bound charge. Note that we should consider either $\mathbf{P}$ or σ_b and ρ_b, but not both, since they are equivalent representation of the same effect. The two equations serve simply for conversions.

2.2 Electric Displacement

In dielectric, we have

$$\rho = \rho_f + \rho_b$$

where f stands for free charge. So the Gauss's law reads

$$\epsilon_0 \boldsymbol{\nabla} \cdot \mathbf{E} = \rho = \rho_b + \rho_f = -\boldsymbol{\nabla} \cdot \mathbf{P} + \rho_f$$

therefore,

$$\boldsymbol{\nabla} \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_f$$

Define the electric displacement $\mathbf{D}$ by

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

Then

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f$$

Note that $\mathbf{D}$ is known as long as we know the free charge distribution. The problem of bound charges (are hence dielectric) are not present. Also note that in Gauss's law $\mathbf{E}$ is always the total field, and in this case there is no such thing as an induced field. The boundary conditions for $\mathbf{D}$ are simple,

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = 0$$

$$\mathbf{D}^{\parallel}_{\text{above}} - \mathbf{D}^{\parallel}_{\text{below}} = \mathbf{P}^{\parallel}_{\text{above}} - \mathbf{P}^{\parallel}_{\text{below}}$$

2.3 Linear Dielectrics

For linear dielectrics,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

So

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}$$

where we define the permittivity

$$\epsilon = \epsilon_0 (1 + \chi_e)$$

We also define

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

3 Magnetostatics

3.1 Magnetic Force

Magnetic force is given by

$$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$$

For a current, we have

$$\mathbf{F} = \int (\mathbf{v} \times \mathbf{B}) dq = \int (\mathbf{v} \times \mathbf{B}) \lambda dl = \int I(d\mathbf{l} \times \mathbf{B})$$

The surface and volume charge densities are given by

$$\mathbf{K} = \sigma \mathbf{v} \qquad \mathbf{J} = \rho \mathbf{v}$$

The continuity equation for charge is

$$\boldsymbol{\nabla} \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

3.2 The Biot-Savart Law

The magnetic field of a steady line current is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl'$$

Note that for a straight wire, **B** is always perpendicular to the wire.

3.3 Divergence and Curl of B

One form of the Biot-Savart law is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau'$$

Using a vector identity, we can write

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \frac{\mu_0}{4\pi} \int \left[\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\boldsymbol{\nabla} \times \mathbf{J}) - \mathbf{J} \cdot \left(\boldsymbol{\nabla} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) \right] d\tau'$$

J is a property of the distribution, which means $\mathbf{J} = \mathbf{J}(\mathbf{r}')$. So the first term is 0. We can prove that $\boldsymbol{\nabla} \times (r^n \hat{\mathbf{r}}) = \mathbf{0}$, so the second term is also 0. Therefore,

$$\boldsymbol{\nabla} \cdot \mathbf{B} = \mathbf{0}$$

Using a different vector identity, we have

$$\boldsymbol{\nabla} \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \left[\left(\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot \boldsymbol{\nabla} \right) \mathbf{J} - (\mathbf{J} \cdot \boldsymbol{\nabla}) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} + \mathbf{J} \left(\boldsymbol{\nabla} \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} (\boldsymbol{\nabla} \cdot \mathbf{J}) \right] d\tau'$$

which simplifies to

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \left[\mathbf{J} \left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \right] d\tau'$$

We can prove that

$$\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 4\pi \delta^3(\boldsymbol{\mathscr{r}})$$

For the second term, note that

$$(\mathbf{J} \cdot \nabla) \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = -(\mathbf{J} \cdot \nabla') \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

The point here is that we want to be able to manipulate $\mathbf{J}$ to allow some tricks. Then we consider each component of $\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$ and use vector identities, and finally we can prove that

$$(\mathbf{J} \cdot \nabla') \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = 0$$

Therefore,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

3.4 Ampère's Law

Using Stokes' theorem, we get

$$\oint \mathbf{B} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a} = \mu_0 I_{\text{enc}}$$

which is called Ampère's law.

3.5 Magnetic Vector Potential

The vector potential $\mathbf{A}$ satisfies

$$\mathbf{B} = \nabla \times \mathbf{A}$$

There is still freedom in determining $\mathbf{A}$. It is proved that for any $\mathbf{B}$, we can always pick an $\mathbf{A}$ such that

$$\nabla \cdot \mathbf{A} = 0$$

Substitute $\mathbf{A}$ into Ampère's law, we get

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$$

This is nothing but three Poisson's equations. If J is 0 at infinity, the solution is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}}{\mathscr{r}} d\tau'$$

The boundary conditions for $\mathbf{B}$ are

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}$$
$$B^{\parallel}_{\text{above}} - B^{\parallel}_{\text{below}} = \mu_0 K$$

4 Magnetic Fields in Matter

4.1 Magnetization

Magnetic dipole moment $\mathbf{m}$ is defined for a closed current loop by

$$\mathbf{m} = I \int d\mathbf{a}$$

Magnetization $\mathbf{M}$ is the magnetic dipole moment per unit volume. So

$$\mathbf{m} = \int \mathbf{M} d\tau$$

Just like V, we can do a multipole expansion for $\mathbf{A}$. The dipole term is

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

Using the same procedure of V, we arrive at

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{1}{\imath} (\boldsymbol{\nabla}' \times \mathbf{M}) d\tau' + \frac{\mu_0}{4\pi} \oint \frac{1}{\imath} (\mathbf{M} \times d\mathbf{a}')$$

Just like ρ is the source of V as Poisson's equation indicates, $\mathbf{J}$ is the source of $\mathbf{A}$. Comparing expressions of $\mathbf{A}$, we get

$$\mathbf{J}_b = \boldsymbol{\nabla}' \times \mathbf{M} \qquad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

4.2 The Auxiliary Field H

Again similar to ρ, we can write

$$\mathbf{J} = \mathbf{J}_b + \mathbf{J}_f$$

Ampère's law reads

$$\frac{1}{\mu_0} \boldsymbol{\nabla} \times \mathbf{B} = \mathbf{J}_b + \mathbf{J}_f = \boldsymbol{\nabla} \times \mathbf{M} + \mathbf{J}_f$$

So

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \right) = \mathbf{J}_f$$

Define

$$\mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

Then Ampère's law becomes

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f$$

4.3 Linear Media

In linear media, we have

$$\mathbf{M} = \chi_m \mathbf{H}$$

So

$$\mathbf{B} = \mu_0 (1 + \chi_m) \mathbf{H} = \mu \mathbf{H}$$

5 Electrodynamics

5.1 Electromotive Force

5.1.1 Ohm's Law

In most cases, the current density is proportional to the force per unit charge. That is

$$\mathbf{J} = \sigma \mathbf{f}$$

where σ is called conductivity and material-dependent. $\rho = 1/\sigma$ is called resistivity.
$\mathbf{f}$ is normally electromagnetic force, so

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Assume that the second term is small, which is true for many applications, we get

$$\mathbf{J} = \sigma \mathbf{E}$$

This is Ohm's law. From this we can derive some more familiar equations, such as $V = IR$, $P = I^2 R$, and so on. Note that effects such as collisions are taken care by σ, and $\mathbf{E}$ is irrelevant to those effects.

We have

$$\nabla \cdot \mathbf{E} = \frac{1}{\sigma} \nabla \cdot \mathbf{J} = -\frac{1}{\sigma} \frac{\partial \rho}{\partial t}$$

For steady current and uniform conductivity,

$$\nabla \cdot \mathbf{E} = 0$$

hence we can use electrostatic tools for $\mathbf{E}$, even though there is a current.
Note that we require steady current and uniform σ.

5.1.2 Electromotive Force

When there is a source (eg. battery) in the circuit,

$$\mathbf{f} = \mathbf{f}_s + \mathbf{E}$$

where $\mathbf{f}_s$ is the force due to the source, for example chemical force. And $\mathbf{E}$ is the electrostatic field.
Note that $\mathbf{E}$ affects the whole circuit, while $\mathbf{f}_s$ is confined in the source.
The electromotive force is defined by

$$\mathcal{E} = \oint \mathbf{f} \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l}$$

For an ideal source of emf, $\sigma \to \infty$. So a finite $\mathbf{J}$ means $\mathbf{f} = \mathbf{0}$, which indicates that inside the source

$$\mathbf{f}_s = -\mathbf{E}$$

Suppose the two nodes of the source are a and b. Then the potential difference is

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{f}_s \cdot d\mathbf{l} = \oint \mathbf{f}_s \cdot d\mathbf{l} = \mathcal{E}$$

5.1.3 Motional emf

We can get motional emf by moving a wire through a magnetic field. Suppose a wire of length h is moving in a $\mathbf{B}$ field with $\mathbf{v}$, then

$$\mathcal{E} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = vBh$$

Let Φ be the flux of $\mathbf{B}$ through the loop, that is

$$\Phi = \int \mathbf{B} \cdot d\mathbf{a}$$

Then we can prove that

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

5.2 Electromagnetic Induction

5.2.1 Faraday's Law

According to Faraday's experiments,

1. if $\mathbf{B}$ field is fixed and we move the loop, there is an emf,
2. if the loop is fixed and $\mathbf{B}$ field is moving, there is an emf,
3. if both field and loop are fixed but the strength is changed, there is an emf.

For the last 2 cases no magnetic force is present, since $\mathbf{v} = \mathbf{0}$. So there has to be an electric field. As experimentally verified, in all 3 cases the emf satisfies

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

So for the last 2 case, we should have

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}$$

where the integration of $d\mathbf{a}$ is over the whole loop, not just the part covered by $\mathbf{B}$.
Using Stokes' theorem, we get

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is Faraday's law. Note that this reduces to $\nabla \times \mathbf{E} = \mathbf{0}$ in the static case.

To get the direction of the induced current correct, we apply Lenz's law, which says that the induced current flows in such a direction that the flux it produces tends to cancel the change.

5.2.2 Inductance

Suppose we have two loops, and loop 1 has a current I_1. According to Biot-Savart law, the flux through loop 2 is proportional to I_1, that is

$$\Phi_2 = M_{21} I_1$$

We have

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\boldsymbol{\nabla} \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2$$

Since $\mathbf{B}_1$ vanishes at infinity, we have

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}}$$

Hence

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

which means

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}$$

Evidently,

$$M_{21} = M_{12} \, (= M)$$

So as long as $I_1 = I_2$, Φ_2 (when I_1 flows) is same to Φ_1 (when I_2 flows).

Now suppose I_1 flows. The flux Φ_1 is also proportional to I_1, so

$$\Phi_1 = L I_1$$

where L is called inductance. Every time I_1 changes, Φ_2 changes, so there is an $\mathcal{E}_2$ in loop 2. Φ_1 also changes, so there is an $\mathcal{E}_1$ in loop 1 too. We have

$$\mathcal{E}_1 = -L \frac{dI_1}{dt}$$

5.2.3 Energy in Magnetic Fields

If we start from no current and build it up to I, work has to be done against the emf. The work done is

$$W = \frac{1}{2} L I^2$$

Starting from here, we can derive that the energy stored in a magnetic field is

$$W = \frac{1}{2\mu_0} \int B^2 d\tau$$

5.3 Maxwell's Equations

5.3.1 Maxwell's Equations

Now we have the following 4 equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

The problem arises when we consider divergence of the last equation.

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 \nabla \cdot \mathbf{J}$$

The left part is always 0, but the right part, in general, is not. To solve the problem, we apply the continuity equation,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{dt} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

So it makes sense to make the following change,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

This means a changing electric field also induces a magnetic field. We define displacement current $\mathbf{J}_d$ by

$$\mathbf{J}_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

Finally, we get Maxwell's equations,

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

5.3.2 Maxwell's Equations in Matter

We want to rewrite the 4 Maxwell's equations for use in matter. Now, we have the bound charge and bound current,

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

When things become nonstatic, a change in ρ_b, hence $\mathbf{P}$, could lead to current. If we consider a rod with $\pm\sigma$ on each end surface (area dA), we have

$$dI = \frac{\partial \sigma}{\partial t} dA = \frac{\partial P}{\partial t} dA$$

So we define polarization current,

$$\mathbf{J}_p = \frac{\partial \mathbf{P}}{\partial t}$$

Note that the continuity equation is satisfied,

$$\boldsymbol{\nabla} \cdot \mathbf{J}_p = \frac{\partial}{\partial t}(\boldsymbol{\nabla} \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}$$

A changing **M**, however, leads to nothing new but a changing $\mathbf{J}_b$, which we have already considered.

With the modification, we have

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \boldsymbol{\nabla} \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

or

$$\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

So Maxwell's equations in matter can be written,

$$\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}$$

Note that we have to keep **E** and **B**, which means to use these equations we need knowledge about conversions. In this context, the displacement current is

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

6 Conservation Laws

The total energy stored in EM field, per unit volume, is given by (as shown in previous sections)

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$$

Now suppose we have a charge distribution that produces **E** and **B** at t. At $t+dt$, the fields will change the charge distribution and do work. For a charge q, the work done by the fields is

$$dW = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\cdot\mathbf{v}dt = q\mathbf{E}\cdot\mathbf{v}dt$$

Recall that $\mathbf{J} = \rho\mathbf{v}$, so for all charges in a volume $\mathcal{V}$, we have

$$\frac{dW}{dt} = \int_{\mathcal{V}} \mathbf{E}\cdot\mathbf{J}d\tau$$

We want to express **J** in terms of the field (**B**). Using Maxwell's equations, we have

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t}$$

Using one of the vector identities, we get

$$\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) = \boldsymbol{\nabla}\cdot(\mathbf{B}\times\mathbf{E}) + \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{E}) = \boldsymbol{\nabla}\cdot(\mathbf{B}\times\mathbf{E}) - \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}$$

We also have

$$\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(E^2) \qquad \mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(B^2)$$

So

$$\mathbf{E}\cdot\mathbf{J} = \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\mathbf{B}\times\mathbf{E}) - \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} - \epsilon_0\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} = -\frac{1}{2}\frac{\partial}{\partial t}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right) - \frac{1}{\mu_0}\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{B})$$

Therefore, using the divergence theorem,

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)d\tau - \frac{1}{\mu_0}\oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{a}$$

This is Poynting's theorem. Remember that W is the work done by fields, not the energy stored in them. The 1st term is the change of energy stored in $\mathcal{V}$; the 2nd term is evidently the energy transported out of $\mathcal{V}$. Suppose that the charge distribution is unchanged but the EM energy in $\mathcal{V}$ increases, that is, LHS is 0 and the 1st term on RHS is negative. Physically, there must be some energy that is transported into $\mathcal{V}$; mathematically, the 2nd term on RHS must be positive. If we define the Poynting vector,

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B})$$

we see that

$$\oint_S (\mathbf{S} \times \mathbf{B}) \cdot d\mathbf{a}$$

needs to be negative, which means **S** is in general pointing towards $\mathcal{V}$ (energy also needs to flow in this direction for the energy in $\mathcal{V}$ to increase). So we see that **S** is the energy flux (per unit time) density (per unit area).

With Poynting vector, we can rewrite Poynting's theorem as

$$\frac{dW}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} u d\tau - \oint_{\mathcal{S}} \mathbf{S} \cdot d\mathbf{a}$$

If there is no work done to charges (eg. empty space), we have

$$\boldsymbol{\nabla} \cdot \mathbf{S} = -\frac{\partial u}{\partial t}$$

which is the continuity equation for energy.

7 Electromagnetic Waves

7.1 1D Waves

Mathematically, a wave is any function f such that

$$f(x,t) = g(x - vt)$$

We can use Newton's second law to examine a rope to get the wave equation,

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Evidently, all $f = g(x - vt)$ satisfies the wave equation. But due to the square in the equation, there is another type of solutions, which is simply $h(x + vt)$. So the general solution to the wave equation is

$$f(x,t) = g(x - vt) + h(x + vt)$$

7.2 Electromagnetic Waves in Vacuum

In vacuum, Maxwell's equations are

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 0 \qquad \boldsymbol{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \qquad \boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

We take the curl of the right column,

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{E}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E} = -\frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B}) = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\boldsymbol{\nabla} \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

That is,

$$\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} \qquad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

Referring to the wave equation, we see that in vacuum the speed of electromagnetic waves is

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = c = 299,792,458\,\mathrm{m/s}$$

This is where people start to realize that light is possibly an EM wave.

Now suppose we have a monochromatic plane wave propagating in x direction. Then (ignoring relative phase)

$$\mathbf{E} = \mathbf{E}_0 e^{i(kx - \omega t)} \qquad \mathbf{B} = \mathbf{B}_0 e^{i(kx - \omega t)}$$

But Maxwell's equations put more constraints on the waves. Since $\nabla \cdot \mathbf{E} = 0$ and $\nabla \cdot \mathbf{B} = 0$, we have

$$ik\mathbf{E} \cdot \hat{\mathbf{x}} = 0 \qquad ik\mathbf{B} \cdot \hat{\mathbf{x}} = 0$$

which means

$$E_x = 0 \qquad B_x = 0$$

That is, in vacuum, electromagnetic waves are transverse: **E** and **B** are always perpendicular to the propagation direction. In addition, from $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, we have

$$-\frac{\partial E_z}{\partial x}\hat{\mathbf{y}} + \frac{\partial E_y}{\partial x}\hat{\mathbf{z}} = i\omega \mathbf{B}$$

or more compactly,

$$\mathbf{B} = \frac{k}{\omega}(\hat{\mathbf{x}} \times \mathbf{E}) = \frac{1}{c}(\hat{\mathbf{k}} \times \mathbf{E})$$

This means **E** and **B** are in phase and mutually perpendicular.

The energy flow of EM waves is still governed by Poynting vector **S**.

7.3 Electromagnetic Waves in Matter

Suppose

1. there is no free charge or current,
2. the material is linear, so $\mathbf{D} = \epsilon \mathbf{E}$, $\mathbf{B} = \mu \mathbf{H}$,
3. the material is homogeneous, so that μ and ϵ are constants.

Then matter-form Maxwell's equations reduce to

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu\epsilon\frac{\partial \mathbf{E}}{\partial t}$$

Note that now $\mu_0\epsilon_0$ is replaced by $\mu\epsilon$. So the wave speed is

$$v = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n}$$

where the index of refraction n is defined by

$$n = \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

All previous conclusions for vacuum waves hold for EM waves in matter, which the trivial replacements $\mu_0 \to \mu$, $\epsilon_0 \to \epsilon$.