Tensor

- Let $\mathcal{L}(\mathcal{V}_M, \mathcal{W}_N)$ denote the vector space of all linear transformations from $\mathcal{V}$ to $\mathcal{W}$. Let $\mathcal{M}_{N\times M}$ denote the vector space of all $N \times M$ matrices. Then $\mathcal{L}(\mathcal{V}_M, \mathcal{W}_N) \cong \mathcal{M}_{N\times M}$. So $\mathcal{L}(\mathcal{V}_M, \mathcal{W}_N)$ has dimension $M \cdot N$.

- The *dual space* $\mathcal{V}^*$ of a vector space $\mathcal{V}$ is the space $\mathcal{L}(\mathcal{V}, \mathbb{R})$. And we have $\dim(\mathcal{V}^*) = \dim(\mathcal{V})$.

- Suppose $\{\mathbf{e}_i\}_{i=1}^N$ is a basis for the vector space $\mathcal{V}_N$. The *dual* of this basis is $\{\boldsymbol{\epsilon}^j\}_{j=1}^N$, which is a basis for $\mathcal{V}^*$, and has the property $\boldsymbol{\epsilon}^j(\mathbf{e}_i) = \delta_i^j$.

- A map $\mathbf{T} : \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_r \to \mathcal{W}$ is called r*-linear* if it is linear in all its variables.

- Let $\boldsymbol{\tau}_1 \in \mathcal{V}_1^*$ and $\boldsymbol{\tau}_2 \in \mathcal{V}_2^*$. Then we can construct a bilinear map $\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2 : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{R}$ by
$$\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2(\mathbf{v}_1, \mathbf{v}_2) = \boldsymbol{\tau}_1(\mathbf{v}_1)\boldsymbol{\tau}_2(\mathbf{v}_2)$$
The expression $\boldsymbol{\tau}_1 \otimes \boldsymbol{\tau}_2$ is called the *tensor product* of $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$.

- Let $\mathbf{v} \in \mathcal{V}$. We define the mapping $\mathbf{v} : \mathcal{V}^* \to \mathbb{R}$ by
$$\mathbf{v}(\boldsymbol{\tau}) = \boldsymbol{\tau}(\mathbf{v})$$

- The bilinear map $\mathbf{h} : \mathcal{V}^* \times \mathcal{V} \to \mathbb{R}$ defined by $\mathbf{h}(\boldsymbol{\tau}, \mathbf{v}) = \boldsymbol{\tau}(\mathbf{v})$ is called the *natural pairing* of $\mathcal{V}$ and $\mathcal{V}^*$ into $\mathbb{R}$. It is denoted by $\mathbf{h}(\boldsymbol{\tau}, \mathbf{v}) = \boldsymbol{\tau}(\mathbf{v}) = \langle \boldsymbol{\tau} | \mathbf{v} \rangle$.

- Let $\mathcal{V}$ be a vector space with dual space $\mathcal{V}^*$. Then a *tensor of type* (r, s) is a multilinear mapping
$$\mathbf{T}_s^r : \underbrace{\mathcal{V}^* \times \mathcal{V}^* \cdots \times \mathcal{V}^*}_{r \text{ times}} \times \underbrace{\mathcal{V} \times \mathcal{V} \cdots \times \mathcal{V}}_{s \text{ times}} \to \mathbb{R}$$
All these tensors form a vector space, which is denoted by $\mathcal{T}_s^r(\mathcal{V})$. r is called the *contravariant* degree, and s is called the *covariant* degree.

- A tensor of type $(0, 0)$ is defined to be a scalar; a tensor of type $(1, 0)$ is a vector; a tensor of type $(0, 1)$ is a dual vector.