Mathematical Analysis

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1 The Real and Complex Number Systems

1.1 Ordered Sets

1. An *ordered set* is a set on which an order, denoted by $<$, is defined.

2. Suppose S is an ordered set and $E \subset S$, if there exists $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, then we say that E is *bounded above*, and β is an *upper bound* of E. Definitions of bounded below and lower bound are similar.

3. Suppose S is an ordered set and $E \subset S$ is bounded above. If

   (a) α is an upper bound of E,

   (b) if $\gamma < \alpha$ then γ is not an upper bound of E,

   Then we say that α is the *least upper bound* of E, and we write $\alpha = \sup E$. Similarly, the *greatest lower bound* can be defined, and we write $\alpha = \inf E$.

4. An ordered set S is said to have the *least upper bound property* if
   if (1) $E \subset S$, (2) E is not empty, (3) E is bounded above, then $\sup E$ exists in S.

5. Suppose S is an ordered set with the least upper bound property, and $B \subset S$, B is not empty, B is bounded below. Let L be the set of all lower bounds of B, then $\alpha = \sup L$ exists, and $\alpha = \inf B$.

1.2 Fields

1. An *ordered field* is a field that is also an ordered set, such that

   (a) $x + y < x + z$ if $x, y, z \in F$ and $y < z$,

   (b) $xy > 0$ if $x, y \in F$, $x > 0$, $y > 0$.

2. From the field axioms and 6, all the familiar rules for inequalities can be derived.

1.3 The Real Field

1. There exists an ordered field R which has the least upper bound property, and R contains Q as a subfield.

2. If $x, y \in R$, and $x > 0$, then there exists a positive integer n such that $nx > y$.
   If $x, y \in R$, and $x < y$, then there exists $p \in Q$ such that $x < p < y$.

3. For every real $x > 0$ and every integer $n > 0$ there is one and only one real y such that $y^n = x$.

1.4 The Extended Real Number System

1. The extended real number system consists of the real field R and two symbols, $-\infty$ and $+\infty$. If a subset $E \subset R$ is nonempty and is not bounded above, then $+\infty = \sup E$ in the extended real number system.

1.5 The Complex Field

1. A *complex number* is an ordered pair (a, b) of real numbers. Suppose $x = (a, b)$ and $y = (c, d)$, then we define
   (a) $x + y = (a + c, b + d)$
   (b) $xy = (ac - bd, ad + bc)$
2. The above definitions turn the set of all complex numbers into a field, with identity $(0, 0)$ and unity $(1, 0)$.
3. $i = (0, 1)$
4. If z is a complex number, its *absolute value* $|z|$ is the non-negative square root of $z\bar{z}$; that is, $|z| = \sqrt{z\bar{z}}$.
5. If $a_1, \dots, a_n$ and $b_1, \dots, b_n$ are complex numbers, then

$$\left|\sum_{j=1}^{n} a_j \bar{b}_j\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{l=1}^{n} |b_l|^2\right)$$

1.6 Euclidean Spaces

1. The familiar vector space $\mathbb{R}^k$ with the normal inner product and norm defined is called *Euclidean k-space.*

2 Basic Topology

2.1 Finite, Countable, and Uncountable Sets

1. We write $A \sim B$ if sets A and B have the same cardinality. Let J_n be the set $\{1, 2, \ldots, n\}$, and J the set of all positive integers. For any set A, we say
   (a) A is *finite* if $A \sim J_n$ for some n.
   (b) A is *infinite* if A is not finite.
   (c) A is *countable* if $A \sim J$. (Note that here 'countable' really means 'countably infinite')
   (d) A is *uncountable* if A is neither finite nor countable.
   (e) A is *at most countable* if A is finite or countable.
2. A *sequence* is a function f defined on the set J of all positive integers. The sequence is denoted by $\{x_n\}$ where $x_n = f(n)$. If A is a set and $x_n \in A$ for all $n \in J$, then $\{x_n\}$ is said to be a sequence in A.
3. Every infinite subset of a countable set is countable.
4. Let $\{E_n\}$, $n = 1, 2, 3, \ldots$, be a sequence of countable sets. Then the set $S = \bigcup_{n=1}^{\infty} E_n$ is countable.
5. Let A be a countable set, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$, where each $a \in A$ and need not be distinct. Then B_n is countable.
6. Let A be the set of all sequences whose elements are the digits 0 and 1, then A is uncountable.

2.2 Metric Spaces

1. A set X is a *metric space* if a function $d : X \times X \to \mathbb{R}$ is defined such that
   (1) $d(p, q) \geq 0$, equal only when $p = q$.
   (2) $d(p, q) = d(q, p)$
   (3) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$.

   The elements of X are called *points*. Any function d with these three properties is called a *metric*. Note that every subset of a metric space is also a metric space.
2. A *segment* (a, b) is the set of all real numbers x such that $a < x < b$.
3. An *interval* $[a, b]$ is the set of all real numbers x such that $a \leq x \leq b$.
4. If $a_i < b_i$ for all $i = 1, \ldots, k$, then the set of points $\mathbf{x} = (x_1, \ldots, x_k)$, where $a_i \leq x_i \leq b_i$, is called a k-*cell*. Note that a 2-cell is a rectangle.

5. If $\mathbf{x} \in \mathbb{R}^k$ and $r > 0$, then the *open ball* B with center at $\mathbf{x}$ and radius r is the set of all $y \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{x}| < r$. A *closed ball* is $|\mathbf{y} - \mathbf{x}| \leq r$.

6. A subset $E \subset \mathbb{R}^k$ is *convex* if for any $\mathbf{x}, \mathbf{y} \in E$ and $0 < \lambda < 1$,

   $$\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} \in E$$

   Note that all balls and k-cells are convex.

7. Let (X, d) be a metric space.
   (a) A *neighborhood* of a point p is a set $N_r(p)$ consisting of all points q such that $d(p, q) < r$.
   (b) A point p is a *limit point* if every neighborhood of p contains a point $q \neq p$ such that $q \in E$.
   (c) If $p \in E$ and p is not a limit point of E, then p is an *isolated point.*
   (d) E is *closed* if every limit point of E is a point of E.
   (e) A point p is an *interior point* of E if there is a neighborhood N of p such that $N \subset E$.
   (f) E is *open* if every point of E is an interior point.
   (g) E is *perfect* if every limit point of E is a point of E (closed) and every point of E is a limit point of E (no isolated point).
   (h) E is *bounded* if there is a real number M and a point $q \in X$ such that $d(p, q) < M$ for all $p \in E$.
   (i) E is *dense in* X if every point of X is a limit point of E, or a point of E, or both.

8. Every neighborhood is an open set.

9. If p is a limit point of a set E, then every neighborhood of p contains infinitely many points of E.

10. A finite point set has no limit points.

11. Let $\{E_\alpha\}$ be a (finite or infinite) collection of sets E_α. Then, where c denotes complement,

    $$\left(\bigcup_\alpha E_\alpha\right)^c = \bigcap_\alpha (E_\alpha^c)$$

12. A set F is closed if and only if F^c is open.

13. (a) For any collection $\{G_\alpha\}$ of open sets, $\bigcup_\alpha G_\alpha$ is open.
    (b) For any collection $\{F_\alpha\}$ of closed sets, $\bigcap_\alpha F_\alpha$ is closed.
    (c) For any finite collection $G_1, \ldots, G_n$ of open sets, $\bigcap_{i=1}^n G_i$ is open.
    (d) For any finite collection $F_1, \ldots, F_n$ of closed sets, $\bigcup_{i=1}^n F_i$ is closed.

14. If X is a metric space, $E \subset X$, and X' denotes the set of limit points of E, then the *closure* of E is the set $\overline{E} = E \cup E'$.

15. If X is a metric space and $E \subset X$, then
    (a) $\overline{E}$ is closed.
    (b) $E = \overline{E}$ if and only if E is closed.
    (c) $\overline{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.

16. Let E be a nonempty set of real numbers which is bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. Hence $y \in E$ if E is closed.

17. Suppose $E \subset Y \subset X$. Then E is *open relative to* Y if to each $p \in E$ there is associated an $r > 0$ such that $q \in E$ whenever $d(p, q) < r$ and $q \in Y$.

18. Suppose $Y \subset X$. A subset E of Y is open relative to Y if and only if $E = Y \cap G$ for some open set G of X.

2.3 Compact Sets

1. Let E be a subset of a metric space X. Then an *open cover* of E is a collection $\{G_\alpha\}$ of open sets of X such that $E \subset \bigcup_\alpha G_\alpha$.

2. A subset K of a metric space X is said to be *compact* if every open cover of K contains a finite subcover. That is, if $\{G_\alpha\}$ is an open cover of K, then $K \subset G_{\alpha_1} \cup \cdots \cup G_{\alpha_n}$ for a finite n.

3. Suppose $K \subset Y \subset X$. Then K is compact relative to X if and only if K is compact relative to Y.
   Note that this means we can talk about a compact set without considering its embedding space.

4. Compact subsets of metric spaces are closed.
   Note that the idea of proof is that for any open subset, construct a special open cover such that this open cover does not contain any finite subcover.

5. Closed subsets of compact sets are compact.

6. If F is closed and K is compact, then $F \cap K$ is compact.

7. If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.

8. If $\{K_n\}$ is a sequence of nonempty compact sets such that $K_n \supset K_{n+1}$, then $\bigcap_{n=1}^{\infty} K_n$ is not empty.

9. If E is an infinite subset of a compact set K, then E has a limit point in K.

10. If $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$, such that $I_n \supset I_{n+1}$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.
    Note that the idea of proof is to construct a set E consisting of all a_n (suppose $I_n = [a_n, b_n]$), then show that $\sup E \in I_n$ for every n.
11. Let k be a positive integer. If $\{I_n\}$ is a sequence of k-cells such that $I_n \supset I_{n+1}$, then $\bigcap_{n=1}^{\infty} I_n$ is not empty.
    Note that this is a generalization of the last theorem.
12. Every k-cell is compact.
    Note that the idea of proof is to assume a k-cell is not compact, subdivide the k-cell many times, and consider the property of the sequence of divided k-cells.
13. For a set E in $\mathbb{R}^k$, the following three statements are equivalent:
    (a) E is closed and bounded.
    (b) E is compact.
    (c) Every infinite subset of E has a limit point in E.
14. Every bounded infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

2.4 Perfect Sets

1. Let P be a nonempty perfect set in $\mathbb{R}^k$, then P is uncountable.
2. Every interval $[a, b]$ ($a < b$) is uncountable. In particular, the set of all real numbers is uncountable.
3. The Cantor set contains no segment, but it is perfect (no isolated points), and also uncountable.

2.5 Connected Sets

1. Two subsets A and B of a metric space X are *separated* if both $A \cap \overline{B}$ and $\overline{A} \cap B$ are empty.
2. A set $E \subset X$ is *connected* if E is not a union of two nonempty separated sets.
   Note that separated sets are disjoint, but disjoint sets need not be separated.
3. A subset E of the real line $\mathbb{R}$ is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.

3 Numerical Sequences and Series

3.1 Convergent Sequences

1. A sequence $\{p_n\}$ in a metric space X is said to *converge* if there is a point $p \in X$ with the following property: For every $\varepsilon > 0$, there is an integer N such that $n \geq N$ implies that $d(p_n, p) < \varepsilon$.
   Note that if $\{p_n\}$ converges to p, we write $p_n \to p$ or $\lim_{n\to\infty} p_n = p$.

2. If $\{p_n\}$ does not converge, it is said to *diverge*.

3. The set of all points p_n is the *range* of $\{p_n\}$. The sequence $\{p_n\}$ is *bounded* if its range is bounded. Note that the range of a sequence may be finite or infinite.

4. Let $\{p_n\}$ be a sequence in a metric space X.
   (a) $\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of p contains p_n for all but finitely many n.
   (b) If $p \in X, p' \in X$, and if $\{p_n\}$ converges to p and to p', then $p' = p$.
   (c) If $\{p_n\}$ converges, then $\{p_n\}$ is bounded.
   (d) If $E \subset X$ and if p is a limit point of E, then there is a sequence $\{p_n\}$ in E such that $p = \lim_{n\to\infty} p_n$.

   Note that for (b) we use the triangle inequality for metric space. And for (d), consider points such that $d(p_n, p) < 1/n$.

5. Suppose $\{s_n\}$, $\{t_n\}$ are complex sequences, and $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} t_n = t$. Then
   (a) $\lim_{n\to\infty}(s_n + t_n) = s + t$
   (b) $\lim_{n\to\infty}(cs_n) = cs_n$, $\lim_{n\to\infty}(c + s_n) = c + s_n$, for any number c.
   (c) $\lim_{n\to\infty}(s_n t_n) = st$
   (d) $\lim_{n\to\infty}(1/s_n) = 1/s$, provided $s_n \neq 0$ for all n and $s \neq 0$.

6. (a) Suppose $\mathbf{x}_n \in \mathbb{R}^k$ $(n = 1, 2, 3, \ldots)$ and

$$\mathbf{x}_n = (\alpha_{1,n}, \ldots, \alpha_{k,n})$$

   Then $\{\mathbf{x}_n\}$ converges to $\mathbf{x} = (\alpha_1, \ldots, \alpha_k)$ if and only if

$$\lim_{n\to\infty} \alpha_{j,n} = \alpha_j$$

   (b) Suppose $\{\mathbf{x}_n\}$, $\{\mathbf{y}_n\}$ are sequences in $\mathbb{R}^k$, $\{\beta_n\}$ is a sequence of real numbers, and $\{\mathbf{x}_n\} \to \mathbf{x}$, $\{\mathbf{y}_n\} \to \mathbf{y}$, $\{\beta_n\} \to \beta$. Then

$$\lim_{n\to\infty} (\mathbf{x}_n + \mathbf{y}_n) = \mathbf{x} + \mathbf{y} \qquad \lim_{n\to\infty} (\mathbf{x}_n \cdot \mathbf{y}_n) = \mathbf{x} \cdot \mathbf{y} \qquad \lim_{n\to\infty} (\beta_n \mathbf{x}_n) = \beta \mathbf{x}$$

3.2 Subsequences

1. Given a sequence $\{p_n\}$ consider a sequence $\{n_k\}$ of positive integers, such that $n_1 < n_2 < \dots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$. If $\{p_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{p_n\}$.
2. $\{p_n\}$ converges to p if and only if every subsequence of $\{p_n\}$ converges to p.
3. (a) If $\{p_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{p_n\}$ converges to a point of X.
   Note that we use the fact that an infinite subset of a compact set has a limit point.
   (b) Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.
   Note that this is because a bounded subset is a subset of its closure, which is closed and bounded, and hence compact in $\mathbb{R}^k$.
4. The subsequential limits of a sequence $\{p_n\}$ in a metric space X form a closed subset of X.
   Note that we need to use the fact that every element in the set is a limit of a subsequence, so an arbitrarily close point can be found.

3.3 Cauchy Sequences

1. A sequence $\{p_n\}$ in a metric space X is a *Cauchy sequence* if for every $\varepsilon > 0$, there is an integer N such that $d(p_n, p_m) < \varepsilon$ if $n \geq N$ and $m \geq N$.
2. Let E be a nonempty subset of a metric space X, and let S be the set of all real numbers of the form $d(p, q)$, with $p \in E$ and $q \in E$. Then $\sup S$ is called the *diameter* of E.
3. If $\{p_n\}$ is a sequence in X and E_N is the set consisting of points $p_N, p_{N+1}, \dots$, then $\{p_n\}$ is a Cauchy sequence if and only if
   $$\lim_{N\to\infty} \operatorname{diam} E_N = 0$$
4. (a) If $\overline{E}$ is the closure of a set E in a metric space X, then $\operatorname{diam} \overline{E} = \operatorname{diam} E$.
   (b) If $\{K_n\}$ is a sequence of compact sets in X such that $K_n \supset K_{n+1}$ and if
   $$\lim_{n\to\infty} \operatorname{diam} K_n = 0$$
   Then $\bigcap_{n=1}^{\infty} K_n$ consists of exactly one point.
   Note that if it contains more than one point, the diameter would not be 0.
5. (a) In any metric space X, every convergent sequence is a Cauchy sequence.
   (b) If X is a compact metric space and if $\{p_n\}$ is a Cauchy sequence in X, then $\{p_n\}$ converges to some point in X.
   (c) In $\mathbb{R}^k$, every Cauchy sequence converges.
6. A metric space in which every Cauchy sequence converges is said to be *complete*.
   Note that all compact metric spaces and all Euclidean spaces are complete.

7. Every closed subset E of a complete metric space is complete.
8. A sequence $\{s_n\}$ of $\mathbb{R}$ is said to be
   (a) *monotonically increasing* if $s_n \leq s_{n+1}$ for all n,
   (b) *monotonically decreasing* if $s_n \geq s_{n+1}$ for all n.
9. Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.
   Note that we need to use the sup or inf of the range of $\{s_n\}$.

3.4 Upper and Lower Limits

1. Let $\{s_n\}$ be a sequence of $\mathbb{R}$ with the following property: For every real M there is an integer N such that $n \geq N$ implies $s_n \geq M$. We then write $s_n \to +\infty$. Similarly, if for every real M there is an integer N such that $n \geq N$ implies $s_n \leq M$, we write $s_n \to -\infty$.
2. Let $\{s_n\}$ be a sequence of $\mathbb{R}$. Let E be the set of numbers x (in the extended real number system) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. We write

   $$s^* = \sup E \qquad s_* = \inf E$$

   The numbers s^*, s_* are called the *upper* and *lower limits* of $\{s_n\}$; we use the notation

   $$\limsup_{n\to\infty} s_n = s^* \qquad \liminf_{n\to\infty} s_n = s_*$$

3. Let $\{s_n\}$ be a sequence of real numbers. Let E and s^* have the same meaning as in the last definition. Then s^* has the following two properties:
   (a) $s^* \in E$
   (b) If $x > s^*$, there is an integer N such that $n \geq N$ implies $s_n < x$.

   Moreover, s^* is the only number with the properties (a) and (b). An analogous result is true for s_*.
4. If $s_n \leq t_n$ for $n \geq N$, where N is fixed, then

   $$\liminf_{n\to\infty} s_n \leq \liminf_{n\to\infty} t_n \qquad \limsup_{n\to\infty} s_n \leq \limsup_{n\to\infty} t_n$$

3.5 Some Special Sequences

1. (a) If $p > 0$, then $\lim_{n\to\infty}(1/n^p) = 0$.
   (b) If $p > 0$, then $\lim_{n\to\infty} \sqrt[n]{p} = 1$.
   (c) $\lim_{n\to\infty} \sqrt[n]{n} = 1$
   (d) If $p > 0$ and α is real, then $\lim_{n\to\infty}[n^\alpha/(1+p)^n] = 0$.
   (e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.

   Note that we use the binomial theorem for proof.

3.6 Series

1. Given a sequence $\{a_n\}$, the expression $\sum_{n=1}^{\infty} a_n$ is called an *infinite series*, or just a *series*. With $\{a_n\}$, we associate a sequence $\{s_n\}$, where
$$s_n = \sum_{k=1}^{n} a_k$$
The numbers s_n are called the *partial sums* of the series. If $\{s_n\}$ converges to s, we say that the series *converges*, and write
$$\sum_{n=1}^{\infty} a_n = s$$
s is called the sum of the series. It is the limit of $\{s_n\}$ and is not obtained simply by addition. Note that a series is an infinite sum, and is not a sequence itself. And in the following discussions the series and sequences are complex.

2. $\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer N such that
$$\left| \sum_{k=n}^{m} a_k \right| \leq \varepsilon$$
if $m \geq n \geq N$.
Note that this follows from the definition of Cauchy sequence.

3. If $\sum a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.
Note that this follows from the last theorem by taking $m = n$. The converse of the this theorem is incorrect.

4. A series of non-negative real terms converges if and only if its partial sums form a bounded sequence.
Note that this is because $\{s_n\}$ is monotonic.

5. (a) If $|a_n| \leq c_n$ for $n \geq N_0$, where N_0 is some fixed number, and if $\sum c_n$ converges, then $\sum a_n$ converges.
   (b) If $a_n \geq d_n \geq 0$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.

3.7 Series of Non-negative Terms

1. If $0 \leq x < 1$, then
$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$
If $x \geq 1$, the series diverges.

2. Suppose $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series
$$\sum_{k=0}^{\infty} 2^k a_{2^k} = a_1 + 2a_2 + 4a_4 + 8a_8 + \dots$$
converges.
Note the condition is that $\{a_n\}$ is non-negative and monotonically decreasing.

3. $\sum(1/n^p)$ converges if $p > 1$ and diverges if $p \leq 1$.
Note that we use the last theorem and the convergence of the geometrical series.

4. If $p > 1$, the series
$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$
converges. If $p \leq 1$, the series diverges.

3.8 The Number e

1. $e = \sum_{n=0}^{\infty} \frac{1}{n!}$.
Note that this is the definition of e. The following is a theorem.

2. $\lim_{n\to\infty}(1 + 1/n)^n = e$

3. e is irrational.
Note that the proof uses the property of $e - s_n$, where s_n is the partial sum.

3.9 The Root and Ratio Tests

1. (Root Test) Given $\sum a_n$, put $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$. Then
   (a) If $\alpha < 1$, $\sum a_n$ converges.
   (b) If $\alpha > 1$, $\sum a_n$ diverges.
   (c) If $\alpha = 1$, the test gives no information.

2. (Ratio Test) The series $\sum a_n$
   (a) converges if $\limsup_{n\to\infty} |a_{n+1}/a_n| < 1$.
   (b) diverges if $|a_{n+1}/a_n| \geq 1$ for all $n \geq n_0$, where n_0 is some fixed integer.

   Note that in both tests, we use the fact that if $\limsup_{n\to\infty} < 1$, then there exists β with $\limsup_{n\to\infty} < \beta < 1$, and $|a_n| < \beta$ for $n \geq N$, where N is some fixed integer.

3. For any sequence $\{c_n\}$ of positive numbers,
$$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n}$$
$$\limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}$$

3.10 Power Series

1. Given a sequence $\{c_n\}$ of complex numbers, the series $\sum_{n=0}^{\infty} c_n z^n$ is called a *power series*. The numbers c_n are called the *coefficients* of the series, and z is a complex number.

2. Given the power series $\sum c_n z^n$, put

   $$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|} \qquad R = \frac{1}{\alpha}$$

   Then $\sum c_n z^n$ converges if $|z| < R$, and diverges if $|z| > R$. R is called the radius of convergence of $\sum c_n z^n$.
   Note that this is a direct consequence of the root test.

3.11 Summation by Parts

1. Given two sequences $\{a_n\}$, $\{b_n\}$, put $A_n = \sum_{k=0}^{n} a_k$ for $n \geq 0$, and put $A_{-1} = 0$. Then, if $0 \leq p \leq q$, we have

   $$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p$$

   Note that $a_n = A_n - A_{n-1}$.

2. Suppose

   (a) the partial sums A_n of $\sum a_n$ form a bounded sequence,
   (b) $b_0 \geq b_1 \geq b_2 \geq \dots$,
   (c) $\lim_{n\to\infty} b_n = 0$.

   Then $\sum a_n b_n$ converges.

3. Suppose

   (a) $|c_1| \geq |c_2| \geq |c_3| \geq \dots$,
   (b) $c_{2m-1} \geq 0$, $c_{2m} \leq 0$ $(m = 1, 2, 3, \dots)$,
   (c) $\lim_{n\to\infty} c_n = 0$.

   Then $\sum c_n$ converges.
   Note that we use the last theorem, with $a_n = (-1)^{n+1}$, $b_n = |c_n|$. Series for which (b) holds are called alternating series.

4. Suppose the radius of convergence of $\sum c_n z^n$ is 1, and suppose $c_0 \geq c_1 \geq c_2 \geq \dots$, $\lim_{n\to\infty} c_n = 0$. Then $\sum c_n z^n$ converges at every point on the circle $|z| = 1$, except possibly at $z = 1$.
   Note that if the radius of convergence is not 1, then the convergence at $|z| = 1$ is known and not interesting anymore.

3.12 Absolute Convergence

1. The series $\sum a_n$ is said to *converge absolutely* if the series $\sum |a_n|$ converges. If $\sum a_n$ converges but $\sum |a_n|$ diverges, we say that $\sum a_n$ converges *non-absolutely.*

2. If $\sum a_n$ converges absolutely, then $\sum a_n$ converges.
   Note that the comparison test, the ratio test, and the root test are all for absolute convergence, which then implies convergence.

3.13 Addition and Multiplication of Series

1. If $\sum a_n = A$, and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$, and $\sum(ca_n) = cA$, for any fixed c.

2. Given $\sum a_n$ and $\sum b_n$, we put
   $$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$
   and call $\sum c_n$ the (Cauchy) *product* of the two given series.
   Note that the motivation behind this definition is to collect the coefficients of the terms of same power, when two power series are multiplied term by term.

3. Suppose
   (a) $\sum_{n=0}^{\infty} a_n$ converges absolutely,
   (b) $\sum_{n=0}^{\infty} a_n = A$,
   (c) $\sum_{n=0}^{\infty} b_n = B$,
   (d) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

   Then
   $$\sum_{n=0}^{\infty} c_n = AB$$
   That is, the product of two convergent series converges, and to the right value, if at least one of the two series converges absolutely.

4. If the series $\sum a_n$, $\sum b_n$, $\sum c_n$ converges to A, B, C, and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$, then $C = AB$.
   Note that the proof requires continuity, which is not covered yet.

3.14 Rearrangements

1. Let $\{k_n\}$ be a bijective sequence from J to J, where J denotes the set of positive integers. Putting
   $$a'_n = a_{k_n}$$
   then we say that $\sum a'_n$ is a *rearrangement* of $\sum a_n$.

2. Let $\sum a_n$ be a series of real numbers which converges, but not absolutely. Suppose

   $$-\infty \le \alpha \le \beta \le +\infty$$

   Then there exists a rearrangement $\sum a_n'$ with partial sums s_n' such that

   $$\liminf_{n\to\infty} s_n' = \alpha \qquad \limsup_{n\to\infty} s_n' = \beta$$

   Note that the proof is basically constructing one such rearrangement.

3. If $\sum a_n$ is a series of complex numbers which converges absolutely, then every rearrangement of $\sum a_n$ converges, and they all converge to the same sum.

4 Continuity

4.1 Limits of Functions

1. Let (X, d_X) and (Y, d_Y) be metric spaces. Suppose $E \subset X$, f maps E into Y, and p is a limit point of E. We write $f(x) \to q$ as $x \to p$, or

   $$\lim_{x \to p} f(x) = q$$

   if there is a point $q \in Y$ with the following property: for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

   $$d_Y(f(x), q) < \varepsilon$$

   for all points $x \in E$ for which

   $$0 < d_X(x, p) < \delta$$

   Note that p need not be a point of E, and even if $p \in E$, we may well have $f(p) \neq \lim_{x \to p} f(x)$.

2. Let X, Y, E, f, and p be as in the last definition. Then

   $$\lim_{x \to p} f(x) = q$$

   if and only if

   $$\lim_{n \to \infty} f(p_n) = q$$

   for every sequence $\{p_n\} \in E$ such that

   $$p_n \neq p \qquad \lim_{n \to \infty} p_n = p$$

3. If f has a limit at p, this limit is unique.
   Note that this follows from the uniqueness of limits of sequences.

4. For two complex functions f and g defined on a metric space E. We define $f + g$, $f - g$, fg, and f/g, with the understanding that the quotient is defined only at those points $x \in E$ at which $g(x) \neq 0$.

5. Suppose $E \subset X$, a metric space, p is a limit point of E, f and g are complex functions on E, and

   $$\lim_{x \to p} f(x) = A \qquad \lim_{x \to p} g(x) = B$$

   Then

   (a) $\lim_{x \to p}(f + g)(x) = A + B$
   (b) $\lim_{x \to p}(fg)(x) = AB$
   (c) $\lim_{x \to p}(f/g)(x) = A/B$, if $B \neq 0$

   Note that properties of limits of functions follow from properties of limits of sequences.

4.2 Continuous Functions

1. Suppose X and Y are metric spaces, $E \subset X$, $p \in E$, and f maps E into Y. Then f is *continuous at* p if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that
   $$d_Y(f(x), f(p)) < \varepsilon$$
   for all points $x \in E$ for which $d_X(x, p) < \delta$. If f is continuous at every point of E, then f is *continuous on* E.
   Note that f has to be defined at p in order to be continuous at p. And if p is an isolated point of E, then every f that is defined on E is continuous at p.

2. In the situation given in the last definition, assume further that p is not an isolated point of E (so that p is a limit point of E). Then f is continuous at p if and only if $\lim_{x \to p} f(x) = f(p)$.
   Note that this follows directly from the two related definitions.

3. Suppose X, Y, Z are metric spaces, $E \subset X$, f maps E into Y, g maps the range of f, $f(E)$, into Z, and h is the mapping of E into Z defined by
   $$h(x) = g\left(f(x)\right) = g \circ f(x)$$
   If f is continuous at a point $p \in E$ and g is continuous at the point $f(p)$, then h is continuous at p.

4. A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(V)$ is open in X for every open set V in Y.

5. A mapping f of a metric space X into a metric space Y is continuous on X if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.
   Note that this follows from the last theorem.

6. Let f and g be complex continuous functions on a metric space X. Then $f + g$, fg, and f/g are continuous on X.
   Note that we assume $g(x) \neq 0$ for all $x \in X$, otherwise the last case is incorrect.

7. (a) Let $f_1, \dots, f_k$ be real functions on a metric space X, and let $\mathbf{f}$ be the mapping of X into $\mathbb{R}^k$ defined by
      $$\mathbf{f}(x) = (f_1(x), \dots, f_k(x))$$
      Then $\mathbf{f}$ is continuous if and only if each of the functions $f_1, \dots, f_k$ is continuous.

   (b) If $\mathbf{f}$ and $\mathbf{g}$ are continuous mappings of X into $\mathbb{R}^k$, then $\mathbf{f} + \mathbf{g}$ and $\mathbf{f} \cdot \mathbf{g}$ are continuous on X.

   Note that the proof follows from the last theorem and inequalities regarding vector and its components.

4.3 Continuity and Compactness

1. A mapping **f** of a set E into $\mathbb{R}^k$ is said to be *bounded* if there is a real number M such that $|\mathbf{f}(x)| \leq M$ for all $x \in E$.

2. Suppose f is a continuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact.

3. If f is a continuous mapping of a compact metric space X into $\mathbb{R}^k$, then $\mathbf{f}(X)$ is closed and bounded. So **f** is bounded.
   Note that this follows from the last theorem and property of $\mathbb{R}^k$.

4. Suppose f is a continuous real function on a compact metric space X, and

   $$M = \sup_{x \in X} f(x) \qquad m = \inf_{x \in X} f(x)$$

   Then there exists points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.
   Note that this follows from the last theorem.

5. Suppose f is a continuous bijection from a compact metric space X to a metric space Y. Then the inverse mapping f^{-1} defined on Y by

   $$f^{-1}\left(f(x)\right) = x$$

   is a continuous mapping of Y onto X.

6. Let f be a mapping of a metric space X into a metric space Y. We say that f is *uniformly continuous* on X if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

   $$d_Y\left(f(p), f(q)\right) < \varepsilon$$

   for all p, q in X for which $d_X(p, q) < \delta$.
   Note that in this definition p is not a chosen point, but an arbitrary point. So uniform continuity is stronger than continuity, and every uniformly continuous function is continuous.

7. Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X.

8. Let E be a non-compact set in $\mathbb{R}$. Then

   (a) there exists a continuous function on E which is not bounded.
   (b) there exists a continuous and bounded function on E which has no maximum.

   If, in addition, E is bounded, then

   (c) there exists a continuous function on E which is not uniformly continuous.

4.4 Continuity and Connectedness

1. If f is a continuous mapping of a metric space X into a metric space Y, and if E is connected, then $f(E)$ is connected.

2. Let f be a continuous real function on the interval $[a, b]$. If $f(a) < f(b)$, and if c is a number such that $f(a) < c < f(b)$, then there exists a point $x \in (a, b)$ such that $f(x) = c$.
   Note that this follows from the last theorem.

4.5 Discontinuities

1. Let f be defined on E and let $x \in E$. If f is not continuous at x, we say that f is *discontinuous* at x.

2. Let f be defined on (a, b). Consider any point x such that $a \leq x < b$. We write

   $$f(x+) = q$$

   if $f(t_n) \to q$ as $n \to \infty$, for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. To obtain the definition of $f(x-)$, for $a < x \leq b$, we restrict ourselves to sequences $\{t_n\}$ in (a, x).
   Note that for any $x \in (a, b)$, $\lim_{t \to x} f(t)$ exists if and only if

   $$f(x+) = f(x-) = \lim_{t \to x} f(t)$$

3. Let f be defined on (a, b). If f is discontinuous at a point x, and if $f(x+)$ and $f(x-)$ exist, then f is said to have a discontinuity of the *first kind*, or a *simple discontinuity*, at x. Otherwise the discontinuity is said to be of the *second kind*.

4.6 Monotonic Functions

1. Let f be real on (a, b). Then f is said to be *monotonically increasing* on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. If $f(x) \geq f(y)$, we obtain the definition of a *monotonically decreasing* function.

2. Let f be monotonically increasing on (a, b). Then $f(x+)$ and $f(x-)$ exist at every point of x of (a, b). More precisely,

   $$\sup_{a<t<x} f(t) \leq f(x-) \leq f(x) \leq f(x+) \leq \inf_{x<t<b} f(t)$$

   Furthermore, if $a < x < y < b$, then

   $$f(x+) \leq f(y-)$$

   And analogous results hold for monotonically decreasing functions.

3. Monotonic functions have no discontinuities of the second kind.
   Note that this follows from the last theorem.

4. Let f be monotonic on (a, b). Then the set of points of (a, b) at which f is discontinuous is at most countable.
   Note that the idea is to construct a one-to-one correspondence between the set of discontinuous points and a subset of rational numbers.

4.7 Infinite Limits and Limits at Infinity

1. For any real c, the set of real numbers x such that $x > c$ is called a neighborhood of $+\infty$ and is written $(c, +\infty)$. Similarly, the set $(-\infty, c)$ is a neighborhood of $-\infty$.
2. Let f be a real function defined on $E \subset \mathbb{R}$. We say that

   $$f(t) \to A \text{ as } t \to x$$

   where A and x are in the extended real number system, if for every neighborhood U of A there is a neighborhood V of x such that $V \cap E$ is not empty, and such that $f(t) \in U$ for all $t \in V \cap E$, $t \neq x$.

5 Differentiation

5.1 The Derivative of a Real Function

1. Let f be defined (and real-valued) on $[a, b]$. For any $x \in [a, b]$ form the quotient

   $$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad (a < t < b,\ t \neq x)$$

   and define

   $$f'(x) = \lim_{t \to x} \phi(t)$$

   provided that this limit exists. f' is called the *derivative* of f. If f' is defined at a point x, we say that f is *differentiable* at x.

2. Let f be defined on $[a, b]$. If f is differentiable at a point $x \in [a, b]$, then f is continuous at x.

3. Suppose f and g are defined on $[a, b]$ and are differentiable at a point $x \in [a, b]$. Then $f + g$, fg, and f/g are differentiable at x, and

   (a) $(f + g)'(x) = f'(x) + g'(x)$
   (b) $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
   (c) $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$

   In (c) we assume that $g(x) \neq 0$.

4. Suppose f is continuous on $[a, b]$, $f'(x)$ exists at some point $x \in [a, b]$, g is defined on an interval I which contains the range of f, and g is differentiable at the point $f(x)$. If

   $$h(t) = g\left(f(t)\right) \qquad (a \leq t \leq b)$$

   then h is differentiable at x, and

   $$h'(x) = g'\left(f(x)\right) f'(x)$$

5.2 Mean Value Theorems

1. Let f be defined on a metric space X. We say that f has a *local maximum* at a point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ for all $q \in X$ with $d(p, q) < \delta$. Local minima are defined likewise.

2. Let f be defined on $[a, b]$. If f has a local maximum at a point $x \in (a, b)$, and if $f'(x)$ exists, then $f'(x) = 0$. Similar for local minima.
   Note that the idea is to show $f'(x) \geq 0$ on one side, and $f'(x) \leq 0$ on the other.

3. If f and g are continuous real functions on $[a, b]$ which are differentiable in (a, b), then there is a point $x \in (a, b)$ at which
$$[f(b) - f(a)]\, g'(x) = [g(b) - g(a)]\, f'(x)$$
Note that differentiability is not required at the endpoints.

4. If f is a real continuous function on $[a, b]$ which is differentiable in (a, b), then there is a point $x \in (a, b)$ at which
$$f(b) - f(a) = (b - a)f'(x)$$
Note that this follows from the last theorem by taking $g(x) = x$.

5. Suppose f is differentiable in (a, b).
   (a) If $f'(x) \geq 0$ for all $x \in (a, b)$, then f is monotonically increasing.
   (b) If $f'(x) = 0$ for all $x \in (a, b)$, then f is constant.
   (c) If $f'(x) \leq 0$ for all $x \in (a, b)$, then f is monotonically decreasing.

5.3 The Continuity of Derivatives

1. Suppose f is a real differentiable function on $[a, b]$ and suppose $f'(a) < \lambda < f'(b)$. Then there is a point $x \in (a, b)$ such that $f'(x) = \lambda$.

2. If f is differentiable on $[a, b]$, then f' cannot have any simple discontinuities on $[a, b]$.

5.4 L'Hôpital's Rule

1. Suppose f and g are real and differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$, where $-\infty < a < b < +\infty$. Suppose
$$\frac{f'(x)}{g'(x)} \to A \text{ as } x \to a$$
If
$$f(x) \to 0 \text{ and } g(x) \to 0 \text{ as } x \to a$$
or if
$$g(x) \to +\infty \text{ as } x \to a$$
then
$$\frac{f(x)}{g(x)} \to A \text{ as } x \to a$$
Note that A is in the extended real number system, and it also works if $x \to b$ or $g(x) \to -\infty$. The idea of proof is to show that for any $p > A$, there exists c_1 such that $a < x < c_1$ implies $f(x)/g(x) < p$; and for any $q < A$, there exists c_2 such that $a < x < c_2$ implies $f(x)/g(x) > q$. So $f(x)/g(x) \to A$.

5.5 Derivatives of Higher Order

1. $f^{(n)}$ is called the nth derivative of f.

5.6 Taylor's Theorem

1. Suppose f is a real function on $[a, b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a, b]$, $f^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

   $$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k$$

   Then there exists a point x between α and β such that

   $$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$$

   Note that for $n = 1$ this is the mean value theorem. The proof depends on the mean value theorem as well.

5.7 Differentiation of Vector-valued Functions

1. $\mathbf{f}'(x)$ is the point in $\mathbb{R}^k$ for which

   $$\lim_{t \to x} \left| \frac{\mathbf{f}(t) - \mathbf{f}(x)}{t - x} - \mathbf{f}'(x) \right| = 0$$

   If $f_1, \ldots, f_k$ are the components of $\mathbf{f}$, then

   $$\mathbf{f}' = (f_1', \ldots, f_k')$$

   Note that the mean value theorem and the L'Hôpital's rule are no longer valid (so we cannot use them for complex-valued functions).

2. Suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$ and $\mathbf{f}$ is differentiable in (a, b). Then there exists $x \in (a, b)$ such that

   $$|\mathbf{f}(b) - \mathbf{f}(a)| \leq (b-a)\,|\mathbf{f}'(x)|$$

   Note that the proof uses the Schwarz inequality.

6 The Riemann-Stieltjes Integral

6.1 Definition and Existence of the Integral

1. Let $[a, b]$ be a given interval. By a *partition* P of $[a, b]$ we mean a finite set of points $x_0, x_1, \ldots, x_n$, where
$$a \leq x_0 \leq x_1 \leq \cdots \leq x_n = b$$
We write
$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, 2, \ldots, n)$$
Now suppose f is a bounded real function defined on $[a, b]$. Corresponding to each partition P of $[a, b]$ we put
$$M_i = \sup f(x) \qquad m_i = \inf f(x) \qquad (x_{i-1} \leq x \leq x_i)$$
$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \qquad L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$$
and finally
$$\overline{\int_a^b} f dx = \inf U(P, f) \qquad \underline{\int_a^b} f dx = \sup L(P, f)$$
where the inf and sup are taken over all partitions P of $[a, b]$. The two LHS are called the *upper* and *lower Riemann integrals* of f over $[a, b]$.
If the upper and lower integrals are equal, we say that f is *Riemann-integrable* on $[a, b]$, we write $f \in \mathscr{R}$, and we denote the common value of the two by
$$\int_a^b f dx \qquad \text{or} \qquad \int_a^b f(x) dx$$
This is the *Riemann integral* of f over $[a, b]$. Since f is bounded, there exist two numbers, m and M, such that
$$m \leq f(x) \leq M$$
Hence, for every P,
$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$$
So that the numbers $L(P, f)$ and $U(P, f)$ form a bounded set. This shows that the upper and lower integrals are defined for every bounded function f.

2. Let α be a monotonically increasing function on $[a, b]$. Corresponding to each partition P of $[a, b]$, we write
$$\Delta \alpha_i = \alpha(x_i) - \alpha(x_{i-1})$$
It is clear that $\Delta \alpha_i \geq 0$. For any real function f which is bounded on $[a, b]$, we put
$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta \alpha_i \qquad L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta \alpha_i$$

where M_i, m_i have the same meaning as in the last definition, and we define

$$\overline{\int_a^b} f d\alpha = \inf U(P, f, \alpha) \qquad \underline{\int_a^b} f d\alpha = \sup L(P, f, \alpha)$$

If the two LHS are equal, we denote their common value by

$$\int_a^b f d\alpha \qquad \text{or} \qquad \int_a^b f(x) d\alpha(x)$$

This is the *Riemann-Stieltjes integral* of f with respect to α, over $[a, b]$. If the integral exists, we write $f \in \mathscr{R}(\alpha)$.

3. We say that the partition P^* is a *refinement* of P if $P^* \supset P$. Given two partitions P_1 and P_2, we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.
4. If P^* is a refinement of P, then

   $$L(P, f, \alpha) \leq L(P^*, f, \alpha)$$

   and

   $$U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

   Note that to prove, consider the simple case where P^* has only one more point.
5. $\underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha$
6. $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P such that

   $$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

7. Let E denote the equation

   $$U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$$

   (a) If E holds for some P and some ε, then E holds (with the same ε) for every refinement of P.

   (b) If E holds for $P = \{x_0, \dots, x_n\}$ and if s_i, t_i are arbitrary points in $[x_{i-1}, x_i]$, then

   $$\sum_{i=1}^{n} |f(s_i) - f(t_i)| \, \Delta\alpha_i < \varepsilon$$

   (c) If $f \in \mathscr{R}(\alpha)$ and the hypotheses of (b) hold, then

   $$\left| \sum_{i=1}^{n} f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon$$

8. If f is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$ on $[a,b]$.
   Note that we use the property of f being uniformly continuous.

9. If f is monotonic on $[a,b]$, and if α is continuous on $[a,b]$, then $f \in \mathscr{R}(\alpha)$. (We still assume that α is monotonic.)

10. Suppose f is bounded on $[a,b]$, f has only finitely many points of discontinuity on $[a,b]$, and α is continuous at every point at which f is discontinuous. Then $f \in \mathscr{R}(\alpha)$.
    Note that we divide the sum in two parts: one with segments containing all discontinuous points, and the other one containing all points left in $[a,b]$.

11. Suppose $f \in \mathscr{R}(\alpha)$ on $[a,b]$, $m \le f \le M$, ϕ is continuous on $[m,M]$, and $h(x) = \phi(f(x))$ on $[a,b]$. Then $h \in \mathscr{R}(\alpha)$ on $[a,b]$.

6.2 Properties of the Integral

1. (a) If $f_1, f_2, f \in \mathscr{R}(\alpha)$ on $[a,b]$, then

$$f_1 + f_2 \in \mathscr{R}(\alpha) \qquad cf \in \mathscr{R}(\alpha)$$

for every constant c, and

$$\int_a^b (f_1+f_2)d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha \qquad \int_a^b cf d\alpha = c\int_a^b f d\alpha$$

(b) If $f_1(x) \le f_2(x)$ on $[a,b]$, then

$$\int_a^b f_1 d\alpha \le \int_a^b f_2 d\alpha$$

(c) If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $a < c < b$, then $f \in \mathscr{R}(\alpha)$ on $[a,c]$ and on $[c,b]$, and

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha$$

(d) If $f \in \mathscr{R}(\alpha)$ on $[a,b]$ and if $|f(x)| \le M$ on $[a,b]$, then

$$\left|\int_a^b f d\alpha\right| \le M\left[\alpha(b) - \alpha(a)\right]$$

(e) If $f \in \mathscr{R}(\alpha_1)$ and $f \in \mathscr{R}(\alpha_2)$, then $f \in \mathscr{R}(\alpha_1 + \alpha_2)$ and

$$\int_a^b f d(\alpha_1+\alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2$$

If $f \in \mathscr{R}(\alpha)$ and c is a positive constant, then $f \in \mathscr{R}(c\alpha)$ and

$$\int_a^b f d(c\alpha) = c\int_a^b f d\alpha$$

2. If $f \in \mathscr{R}(\alpha)$ and $g \in \mathscr{R}(\alpha)$ on $[a,b]$, then
    (a) $fg \in \mathscr{R}(\alpha)$
    (b) $|f| \in \mathscr{R}(\alpha)$ and $\left|\int_a^b f d\alpha\right| \leq \int_a^b |f| d\alpha$
3. The *unit step function* I is defined by
$$I(x) = \begin{cases} 0 & (x \leq 0) \\ 1 & (x > 0) \end{cases}$$
4. If $a < s < b$, f is bounded on $[a,b]$, f is continuous at s, and $\alpha(x) = I(x-s)$, then
$$\int_a^b f d\alpha = f(s)$$
5. Suppose $c_n \geq 0$ for $n = 1, 2, 3, \dots$, $\sum c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a,b), and
$$\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$$
Let f be continuous on $[a,b]$. Then
$$\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$$
6. Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a,b]$. Let f be a bounded real function on $[a,b]$. Then $f \in \mathscr{R}(\alpha)$ if and only if $f\alpha' \in \mathscr{R}$. In that case
$$\int_a^b f d\alpha = \int_a^b f(x)\alpha'(x) dx$$
7. Suppose φ is a strictly increasing continuous function that maps an interval $[A,B]$ onto $[a,b]$. Suppose α is monotonically increasing on $[a,b]$ and $f \in \mathscr{R}(\alpha)$ on $[a,b]$. Define β and g on $[A,B]$ given by
$$\beta(y) = \alpha\left(\varphi(y)\right) \qquad g(y) = f\left(\varphi(y)\right)$$
Then $g \in \mathscr{R}(\beta)$ and
$$\int_A^B g d\beta = \int_a^b f d\alpha$$

6.3 Integration and Differentiation

1. Let $f \in \mathscr{R}$ on $[a, b]$. For $a \leq x \leq b$, put

$$F(x) = \int_a^x f(t)dt$$

Then F is continuous on $[a, b]$. Furthermore, if f is continuous at a point x_0 of $[a, b]$, then F is differentiable at x_0, and

$$F'(x_0) = f(x_0)$$

2. If $f \in \mathscr{R}$ on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Note that again we use the mean value theorem to prove.

3. Suppose F and G are differentiable functions on $[a, b]$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$. Then

$$\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$$

Note that for proof we put $H(x) = F(x)G(x)$ and apply the last theorem.

6.4 Integration of Vector-valued Functions

1. Let $f_1, \dots, f_k$ be real functions on $[a, b]$, and let $\mathbf{f} = (f_1, \dots, f_k)$ be the corresponding mapping of $[a, b]$ into $\mathbb{R}^k$. If α increases monotonically on $[a, b]$, to say that $\mathbf{f} \in \mathscr{R}(\alpha)$ means that $f_j \in \mathscr{R}(\alpha)$ for $j = 1, \dots, k$. If this is the case, we define

$$\int_a^b \mathbf{f} d\alpha = \left(\int_a^b f_1 d\alpha, \dots, \int_a^b f_k d\alpha\right)$$

In other words, $\int \mathbf{f} d\alpha$ is the point in $\mathbb{R}^k$ whose jth coordinate is $\int f_j d\alpha$.

2. If $\mathbf{f}$ and $\mathbf{F}$ map $[a, b]$ into $\mathbb{R}^k$, if $\mathbf{f} \in \mathscr{R}$ on $[a, b]$, and if $\mathbf{F}' = \mathbf{f}$, then

$$\int_a^b \mathbf{f}(t)dt = \mathbf{F}(b) - \mathbf{F}(a)$$

3. If $\mathbf{f}$ maps $[a, b]$ into $\mathbb{R}^k$ and if $\mathbf{f} \in \mathscr{R}(\alpha)$ for some monotonically increasing function α on $[a, b]$, then $|\mathbf{f}| \in \mathscr{R}(\alpha)$, and

$$\left|\int_a^b \mathbf{f} d\alpha\right| \leq \int_a^b |\mathbf{f}| d\alpha$$

6.5 Rectifiable Curves

1. A continuous mapping γ of an interval $[a, b]$ into $\mathbb{R}^k$ is called a *curve* in $\mathbb{R}^k$. To emphasize the parameter interval $[a, b]$, we may also say that γ is a curve on $[a, b]$.

    - If γ is one-to-one, γ is called an *arc*.
    - If $\gamma(a) = \gamma(b)$, γ is said to be a *closed curve*.

    We associate to each partition $P = \{x_0, \ldots, x_n\}$ of $[a, b]$ and to each curve γ on $[a, b]$ the number

    $$\Lambda(P, \gamma) = \sum_{i=1}^{n} |\gamma(x_i) - \gamma(x_{i-1})|$$

    The *length* of γ is defined to be

    $$\Lambda(\gamma) = \sup \Lambda(P, \gamma)$$

    where sup is taken over all partitions of $[a, b]$. If $\Lambda(\gamma) < \infty$, we say that γ is *rectifiable*.

2. If γ' is continuous on $[a, b]$, then γ is rectifiable, and

    $$\Lambda(\gamma) = \int_a^b |\gamma'(t)| \, dt$$

7 Sequences and Series of Functions

7.1 Discussion of Main Problem

1. Suppose $\{f_n\}$, $n = 1, 2, 3, \ldots$, is a sequence of functions defined on a set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We can define a function f by

   $$f(x) = \lim_{n\to\infty} f_n(x)$$

   Under these circumstances we say that $\{f_n\}$ converges on E and that f is the *limit*, or the *limit function*, of $\{f_n\}$. Sometimes we shall use a more descriptive terminology and shall say that "$\{f_n\}$ converges to f *pointwise* on E". Similarly, if $\sum f_n(x)$ converges for every $x \in E$, and if we define

   $$f(x) = \sum_{n=1}^{\infty} f_n(x)$$

   the function f is called the *sum* of the series $\sum f_n$.